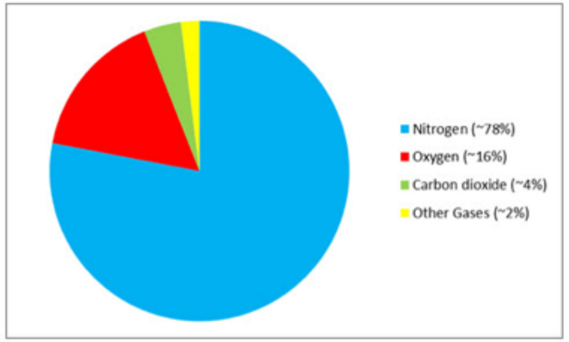

Intégration par parties (IPP)

$$\int_a^b u(x)\,v'(x) = \underbrace{\Big[u(x)\,v(x)\Big]_a^b}_{u(b)v(b)\,-\,u(a)v(a).} - \int_a^b u'(x)\,v(x)\,dx$$

$$\frac{d}{dx}\Big[u(x)\cdot v(x)\Big] = u'(x)\cdot v(x) + u(x)\cdot v'(x)$$

Exemple 9.17 :

(i) Calculons $\int_0^{\pi} \underbrace{(x^2+1)}_{\text{polynôme}} \cdot \underbrace{\cos(x)}\,dx$

on est capable de primitiver autant de fois qu'on veut.

$$= \int_0^{\pi} (x^2+1)\cos(x)\,dx \overset{IPP}{=} \Big[(x^2+1)\cdot\sin(x)\Big]_0^{\pi} - \int_0^{\pi} 2x\sin(x)\,dx$$

$u(x) = x^2+1 \qquad v = \sin(x)$

$u' = 2x \qquad v'(x) = \cos(x)$

$$= \Big((\pi^2+1)\cancel{\sin(\pi)}^{\,0} - (0^2+1)\cdot\cancel{\sin(0)}^{\,0}\Big) - \int_0^{\pi} 2x\sin(x)\,dx$$

$$= -\int_0^{2\pi} 2x\sin(x)\,dx \overset{IPP}{=} -\Big[-2x\cos(x)\Big]_0^{\pi} + \int_0^{\pi} -2\cos(x)$$

$u(x) = 2x \qquad v(x) = -\cos(x)$

$u'(x) = 2 \qquad v'(x) = \sin(x)$

$$= \Big(2\pi\cos(\pi) - 0\cdot\cos(0)\Big) - \int_0^{\pi} 2\cos(x)\,dx$$

$$= -2\pi - \int_0^{\pi} 2\cos(x)\,dx = -2\pi - \Big[2\sin(x)\Big]_0^{\pi}$$

$$= -2\pi - 2\cancel{\sin(\pi)}^{\,0} + 2\cancel{\sin(0)} = -2\pi$$

(ii) Calculons $I = \int_0^\pi \sin(2x)\cdot\cos(3x)\,dx$

$$I = \int_0^\pi \sin(2x)\cos(3x)\,dx \overset{IPP}{=} \left[-\frac{1}{2}\cos(3x)\cos(2x)\right]_0^\pi - \int_0^\pi \frac{3}{2}\cos(2x)\sin(3x)$$

$u(x) = \cos(3x)$ $\quad v = -\frac{1}{2}\cos(2x)$

$u' = -3\sin(3x)$ $\quad v'(x) = \sin(2x)$

$$= -\frac{1}{2}\underbrace{\cos(3\pi)}_{-1}\cdot\underbrace{\cos(2\pi)}_{1} + \frac{1}{2}\underbrace{\cos(0)}_{1}\cdot\underbrace{\cos(0)}_{1}$$

$$-\frac{3}{2}\int_0^\pi \cos(2x)\sin(3x)\,dx$$

$$= 1 - \frac{3}{2}\int_0^\pi \cos(2x)\sin(3x)\,dx \overset{IPP}{=} 1 - \frac{3}{2}\left(\left[\frac{1}{2}\cancelto{0}{\sin(3x)}\cancelto{0}{\sin(2x)}\right]_0^\pi - \int_0^\pi \frac{3}{2}\sin(2x)\cos(3x)\,dx\right)$$

$u(x) = \sin(3x)$ $\quad v(x) = \frac{1}{2}\sin(2x)$

$u'(x) = 3\cos(3x)$ $\quad v'(x) = \cos(2x)$

$$= 1 + \frac{9}{4}\underbrace{\int_0^\pi \sin(2x)\cos(3x)\,dx}_{I}$$

$$\Rightarrow \quad I = 1 + \frac{9}{4} I \quad \Rightarrow \quad 4I = 4 + 9I \quad \Rightarrow \quad 5I = -4 \quad \Rightarrow \quad I = -\frac{4}{5}$$

(iii) Calculons $\displaystyle\int_1^x \log(t)\,dt = \int_1^x 1 \cdot \log(t)\,dt$

$$\boxed{\begin{array}{ll} u(t) = \log(t) & v(t) = t \\ u'(t) = \dfrac{1}{t} & v'(t) = 1 \end{array}}$$

$$\overset{IPP}{=} \big[t \log(t) \big]_1^x - \int_1^x t \cdot \frac{1}{t}\,dt$$

$$= x\log(x) - 1 \cdot \underbrace{\log(1)}_{=0} - \int_1^x 1\,dt$$

$$= x\log(x) - \big[t \big]_1^x = x\log(x) - x + 1.$$

Théorème 9.18 : Changement de variables.

Soient $a<b$, $\alpha<\beta$, $f \in C^0([a,b])$, $\varphi \in C^1([\alpha,\beta])$ tq $\varphi([\alpha,\beta]) \subseteq [a,b]$. Alors,

(i) $\displaystyle\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_\alpha^\beta f(\varphi(x)) \cdot \varphi'(x)\,dx$

(ii) si de plus, $\varphi : [\alpha, \beta] \to [a,b]$ est bijective, et $\varphi^{-1} \in C^1$

$$\int_\alpha^\beta f(\varphi(t))\,dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} \frac{f(x)}{\varphi'(\varphi^{-1}(x))}\,dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\cdot(\varphi^{-1})'(x)\,dx$$

<u>Exemple 9.14</u> : (i) Calculons

$\int_0^1 \frac{1}{(1+x^2)^{3/2}}\,dx$, à l'aide du changement de variables

$\underbrace{x = \tan(t)}$, $t = \arctan(x)$.

$$\frac{dx}{dt} = \frac{1}{\cos^2(t)} \Rightarrow dx = \frac{dt}{\cos^2(t)}.$$

bornes : $x = 0$, $t = ? \leftarrow 0$

$x = 1$, $t = ? \leftarrow \frac{\pi}{4}$

$$= \int_0^{\pi/4} \frac{1}{(1+\tan^2(t))^{3/2}} \cdot \frac{1}{\cos^2(t)}\,dt$$

$$1+\tan^2(t) = \frac{1}{\cos^2(t)}$$

$$= \int_0^{\pi/4} \frac{1}{\left(\frac{1}{\cos^2(t)}\right)^{3/2}} \frac{1}{\cos^2(t)} \, dt$$

$$= \int_0^{\pi/4} \frac{\cos^3(t)}{\cos^2(t)} dt = \int_0^{\pi/4} \cos(t) dt = \Big[\sin(t)\Big]_0^{\pi/4} = \frac{\sqrt{2}}{2}.$$

(ii) $\int_0^3 e^{\sqrt{1+x}} \, dx$

$\boxed{t = \sqrt{1+x}} \rightsquigarrow \boxed{x = t^2 - 1}$

bijectif ? oui !! (dérivée > 0 $\Rightarrow$ injective $\Rightarrow$ bijectif sur son image)

$$\frac{dx}{dt} = 2t \rightsquigarrow dx = 2t\,dt$$

$x = 0, \quad t = ? \leftarrow 1$

$x = 3 \quad t = ? \leftarrow \sqrt{1+3} = 2.$

$$\int_0^3 e^{\sqrt{1+x}}\,dx = \int_1^2 e^t \cdot 2t\,dt$$

$$= 2\int_1^2 t\cdot e^t\,dt \overset{\text{IPP}}{=} 2\left[te^t\right]_1^2 - 2\int_1^2 e^t\,dt$$

$u = t \qquad v = e^t$

$u' = 1 \qquad v' = e^t$

$$= 4e^2 - 2e - 2\left[e^t\right]_1^2$$

$$= 4e^2 - 2e - 2e^2 + 2e = 2e^2$$

Exemple 9.20 (Décomposition en éléments simples)

(i) Calculons $\displaystyle\int_1^3 \frac{1}{x(x^2+1)}\,dx$

$\exists ?\ A, B, C \in \mathbb{R}$ tq $\displaystyle\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$$= \frac{A(x^2+1) + (Bx+C)x}{x(x^2+1)} = \frac{(A+B)x^2 + Cx + A}{x(x^2+1)}.$$

On égalise les numérateurs

$$0 \cdot x^2 + 0 \cdot x + 1 = (A+B)x^2 + C \cdot x + A$$

$$\Rightarrow \begin{cases} A+B = 0 \\ C = 0 \\ A = 1 \end{cases} \Rightarrow A = 1,\ B = -1,\ C = 0$$

$$\Rightarrow \frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}$$

$$\Rightarrow \int_1^3 \frac{1}{x(x^2+1)}\,dx = \int_1^3 \frac{1}{x} - \frac{x}{x^2+1}\,dx = \left[\log|x| - \frac{1}{2}\log(x^2+1) \right]_1^3$$

$$= \log(3) + \frac{1}{2}\log(2) - \frac{1}{2}\log(10)$$

(ii) $\int_a^b \frac{x}{x^2+2x+3}\,dx$ $\qquad x^2+2x+3$ ne se factorise pas.

$$x^2+2x+3 = (x+\alpha)^2+\beta = \underbrace{x^2+2x+1}_{(x+1)^2}-1+3 = (x+1)^2+2$$

compléter le carré

$$= 2\left(\left(\frac{x+1}{\sqrt{2}}\right)^2+1\right)$$

$$\Rightarrow \int_a^b \frac{x}{x^2+2x+3}\,dx = \int_a^b \frac{x}{2\left(\left(\frac{x+1}{\sqrt{2}}\right)^2+1\right)}\,dx = \frac{1}{2}\int_a^b \frac{x}{\left(\frac{x+1}{\sqrt{2}}\right)^2+1}\,dx$$

$$y = \frac{x+1}{\sqrt{2}} \longrightarrow x = \sqrt{2}\,y-1 \rightarrow \frac{dx}{dy} = \sqrt{2} \longrightarrow dx = \sqrt{2}\,dy$$

$x = a,\quad y = \frac{a+1}{\sqrt{2}}$

$x = b,\quad y = \frac{b+1}{\sqrt{2}}$

$$\int_a^b \frac{x}{x^2+2x+3}\,dx = \frac{1}{2}\int_{\frac{a+1}{\sqrt{2}}}^{\frac{b+1}{\sqrt{2}}} \frac{\sqrt{2}\,y-1}{y^2+1}\cdot\sqrt{2}\,dy$$

$$= \int_{\frac{a+1}{\sqrt{2}}}^{\frac{b+1}{\sqrt{2}}} \frac{y}{y^2+1} - \frac{\sqrt{2}}{2}\,\frac{1}{y^2+1}\,dy$$

$$= \left[\frac{1}{2}\log(y^2+1) - \frac{\sqrt{2}}{2}\arctan(y)\right]_{\frac{a+1}{\sqrt{2}}}^{\frac{b+1}{\sqrt{2}}}.$$

(iii) $\displaystyle\int_3^5 \frac{1}{x^3-4x^2+9x-10}\,dx$

$\frac{10}{1} = 10$ $\rightsquigarrow$ on cherche une racine parmis $\pm 1, \pm 2, \pm 5$

2 est une racine de $x^3-4x^2+9x-10 = (x-2)(x^2-2x+5)$

$$\frac{1}{(x-2)(x^2-2x+5)} = \frac{A}{(x-2)} + \frac{Bx+C}{x^2-2x+5}$$

$$= \frac{(A+B)x^2 + (C-2A-2B)x + 5A-2C}{(x-2)(x^2-2x+5)}$$

$$\begin{cases} A+B=0 \\ C-2A-2B=0 \\ 5A-2C=1 \end{cases} \Rightarrow \begin{aligned} A &= \frac{1}{5} \\ B &= -\frac{1}{5} \\ C &= 0 \end{aligned}$$

$$\Rightarrow \frac{1}{(x-2)(x^2-2x+5)} = \frac{1}{5}\frac{1}{x-2} - \frac{1}{5}\frac{x}{x^2-2x+5}$$

Or, $\frac{1}{x^2-2x+5} = \frac{1}{\underbrace{x^2-2x+1}_{(x-1)^2}-1+5} = \frac{1}{(x-1)^2+4} = \frac{1}{4}\cdot\frac{1}{\left(\frac{x-1}{2}\right)^2+1}$

$$\Rightarrow \int_3^5 \frac{1}{x^3-4x^2+9x-10}\,dx = \frac{1}{5}\int_3^5 \frac{1}{x-2}\,dx - \frac{1}{20}\int_3^5 \frac{x}{\left(\frac{x-1}{2}\right)^2+1}\,dx$$

$y = \frac{x-1}{2}$, $x=3$, $y=1$ & $x=5$, $y=2$

$x = 2y+1$, $dx = 2dy$

$$= \frac{1}{5}\Big[\log|x-2|\Big]_3^5 - \frac{1}{20}\int_1^2 \frac{2y+1}{y^2+1}\,2dy$$

$$= \frac{1}{5}\log(3) - \frac{1}{10}\underbrace{\int_1^2 \frac{2y}{y^2+1}}_{\left[\log(y^2+1)\right]_1^2} + \underbrace{\frac{1}{y^2+1}\,dy}_{\left[\arctan(y)\right]_1^2}.$$